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Euclidean and Canonical Formulations of Statistical Mechanics in the Presence of Killing Horizons

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Abstract

The relation between the covariant Euclidean free-energy F^E and the canonical statistical-mechanical free energy F^C in the presence of the Killing horizons is studied. F^E is determined by the covariant Euclidean effective action. The definition of F^C is related to the Hamiltonian which is the generator of the evolution along the Killing time. At arbitrary temperatures F^E acquires additional ultraviolet divergences because of conical singularities. The divergences of F^C are different and occur since the density $\frac{dn}{d\omega}$ of the energy levels of the system blows up near the horizon in an infrared way. We show that there are regularizations that make it possible to remove the infrared cutoff in $\frac{dn}{d\omega}$. After that the divergences of F^C become identical to the divergences of F^E. The latter property turns out to be crucial to reconcile the covariant Euclidean and the canonical formulations of the theory. The method we use is new and is based on a relation between $\frac{dn}{d\omega}$ and heat kernels on hyperbolic-like spaces. Our analysis includes spin 0 and spin 1/2 fields on arbitrary backgrounds. For these fields the divergences of $\frac{dn}{d\omega}$, F^C and F^E are presented in the most complete form.

1 Introduction

There are two approaches how to describe quantum thermal effects in the gravitational field. The approach by Gibbons and Hawking [1],[2] defines the partition function of the system as an Euclidean path integral. It enables one to express the free energy $F^E[g,\beta]$ in terms of the effective action $W[g,\beta]$, as $F^E[g,\beta] = \beta^{-1}W[g,\beta]$. Functionals $W[g,\beta]$ are given on Euclidean manifolds $\mathcal{M}_\beta$ with the period β in the Euclidean time τ. β is considered as the inverse temperature. The fields are assumed to be periodic or antiperiodic in τ, depending on their statistics. The Gibbons-Hawking approach is a straightforward generalization of the finite-temperature theory in the Minkowsky space-time. Its advantage is that it is manifestly covariant in the Euclidean sector and enables one to consider the gravitational field on the equal footing with matter fields. This approach is especially convenient for the thermodynamics of black holes [1]-[5], it reproduces the entropy, temperature and other characteristics of a black hole in the semiclassical approximation.

When the space-time is static statistical-mechanical quantities can be also described in a canonical way. The canonical partition function has the form

$$Z^C = \mathrm{Tr}\ e^{-\beta\hat{\mathcal{H}}} \quad , \tag{1.1}$$

where the Hamiltonian $\hat{\mathcal{H}}$ is the generator of the time evolution of the system. Z^C is well defined when $\hat{\mathcal{H}}$ is a normally ordered operator [6]. In this case the free-energy takes the form

$$F^C[g,\beta] = -\beta^{-1}\ln Z = \eta\beta^{-1}\int_0^\infty d\omega \frac{dn(\omega)}{d\omega}\ln\left(1-\eta e^{-\beta\omega}\right) \quad , \tag{1.2}$$

where $\eta = +1$ for Bose fields and $\eta = -1$ for Fermi fields. $\frac{dn(\omega)}{d\omega}$ is the density of eigenvalues ω of *quantum-mechanical* Hamiltonians of the fields [6]-[14]. The advantage of definition (1.2) is that it is given in accordance with the unitarity evolution of the system. However, as distinct from the Gibbons-Hawking approach, it is not manifestly covariant.

There is no special terminology[1] to distinguish F^E and F^C. In this paper we call F^E and F^C the covariant Euclidean and the canonical free energies, respectively. The corresponding formulations of the finite-temperature theory will be called the covariant Euclidean and the canonical formulations. Sometime we will also say "Euclidean" instead of "covariant Euclidean", for simplicity.

For static space-times without horizons comparing F^E and F^C shows [6] that these functionals differ only by the vacuum energy which is not included in F^C. Thus, in this case the Euclidean and canonical formulations are, in fact, equivalent.

In space-times with Killing horizons the quantum theory has a number of specific properties. On one hand, in the Euclidean formulation there is a distinguished value of the period β, corresponding to the Hawking temperature, for which $\mathcal{M}_\beta$ is a regular

[1] Allen [6] called $\exp(-\beta F^E)$ and $\exp(-\beta F^C)$ the "quantum" and "thermodynamic" partition functions. These names are not very suitable at least because the both partition functions are essentially quantum.

space[2]. At other values of β the space $\mathcal{M}_\beta$ has conical singularities which result in additional ultraviolet divergences [15]-[18]. On another hand, the canonical formulation runs into difficulties because the time evolution is not defined at the bifurcation surface of the Killing horizons. The density $\frac{dn}{d\omega}$ of the energy levels in Eq. (1.2) blows up near the horizon [19]-[21] at any temperature in an infrared way.

As a result, in the presence of horizons the Euclidean and the canonical free energies look different and finding the relation between them becomes a problem. This problem has not been analysed before and our aim is to investigate it for the case of scalar and spinor fields in some details. Comparison of the covariant Euclidean and the canonical formulations is important for different reasons. The main of them is statistical-mechanical interpretation of black hole thermodynamics which can be naturally defined in the framework of the Gibbons-Hawking approach [1]-[5].

The paper is organized as follows. In Section 2 we describe the Euclidean and the canonical formulations of statistical mechanics of scalar and Dirac fields on static backgrounds. The ultraviolet divergences appearing in F^E because of conical singularities are given in Section 3 in the most complete form. We use the dimensional and Pauli-Villars regularization procedures. In Section 4 we develop a method how to find the divergences of the density of levels $\frac{dn}{d\omega}$. We first apply this method to study $\frac{dn}{d\omega}$ in the presence of the spatial cutoff near the horizon. Then we show, in Section 5, that in dimensional and Pauli-Villars regularizations the spatial cutoff can be removed and one can define $\frac{dn}{d\omega}$ on the complete space. In these regularizations the divergences of $\frac{dn}{d\omega}$ have the ultraviolet character. It means that the corresponding divergences of the covariant free energy F^C coincide exactly with the divergences of the Euclidean free energy F^E. In Section 6 we discuss a hypothesis that (for spinors and scalars) the entire bare functionals F^C and F^E must coincide. We illustrate it with some examples. Technical details are given in Appendixes. In Appendix A we remind the reader how to relate the canonical free energy on ultrastatic spaces to the effective action. Appendix B is devoted to the calculation of the spinor heat coefficients on conical singularities, some of the coefficients represent the new result.

2 Definitions and basic relations

Let us consider scalar fields ϕ described by the Klein-Gordon equation and spinor fields ψ described by the Dirac equation,

$$(-\nabla^\mu\nabla_\mu + \xi R + m^2)\phi = 0 \quad , \quad (\gamma^\mu\nabla_\mu + m)\psi = 0 \quad , \tag{2.1}$$

[2]This is the property of nonextremal black hole backgrounds. The extremal black holes will not be considered here.

where R is the scalar curvature and ∇_μ are the covariant derivatives, defined according with the spin of the fields[3]. The Dirac γ-matrices $\gamma^\mu = (\gamma^0, \gamma^a)$ obey the standard commutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, γ^0 is the anti-Hermitean matrix. It is supposed that the space-time is static

$$ds^2 = g_{00}dt^2 + g_{ab}dx^a dx^b \quad , \quad a, b = 1, 2, 3 \quad . \tag{2.2}$$

The component g_{00} is a nonpositive function and $g_{00} = -1$ at spatial infinity. The temperature measured at infinity is β^{-1}. On space (2.2) equations (2.1) can be rewritten in the form

$$-g^{00}(\partial_t^2 + H_s^2)\phi = 0 \quad , \quad -i\gamma^0(i\partial_t + H_d)\psi = 0 \quad , \tag{2.3}$$

$$H_s^2 = |g_{00}|(-\nabla^a\nabla_a - w^a\nabla_a + m^2 + \xi R) \quad , \tag{2.4}$$

$$H_d = i\gamma_0\left(\gamma^a(\nabla_a + \frac{1}{2}w_a) + m\right) \quad . \tag{2.5}$$

Here ∇_a is the covariant derivative computed with the help of the metric g_{ab} of the three-dimensional surface of constant time $t = const$. We denote this surface $\mathcal{B}$. Index a is up and down with the help of g_{ab}. The vector $w_a = \frac{1}{2}\nabla_a \ln|g_{00}|$ is the vector of acceleration. The operators H_s and H_d are called the one-particle Hamiltonians because their eigen-values coincide with the frequencies of one-particle excitations [4]. To calculate the canonical free energies F_i^C $(i = s, d)$ with the help of Eq. (1.2) one has to know the densities $\frac{dn_i}{d\omega}$ of the energy levels of H_s and H_d.

Let us define now the Euclidean free energies for the fields described by Eq. (2.1). To this aim we consider the Euclidean manifold $\mathcal{M}_\beta$ which is the Euclidean section of the Lorentzian geometry (2.2)

$$ds^2 = g_{\tau\tau}d\tau^2 + g_{ab}dx^a dx^b \quad , \quad 0 \leq \tau \leq \beta \quad , \tag{2.6}$$

where $g_{\tau\tau} = |g_{00}|$. The *Euclidean* effective actions for the fields ϕ and ψ are

$$W_s[g, \beta] = \frac{1}{2}\log\det L_s \quad , \quad W_d[g, \beta] = -\log\det L_d \quad , \tag{2.7}$$

$$L_s = -\nabla^\mu\nabla_\mu + \xi R + m^2 \quad , \quad L_d = \gamma_5(\gamma^\mu\nabla_\mu + m) \quad . \tag{2.8}$$

It is assumed that W_i are regularized functionals. The operators L_s act on scalar fields on $\mathcal{M}_\beta$ which are periodic in τ, L_d act on spinors which change the sign when τ is increased by β. The Euclidean matrix γ_τ is $i\gamma_0$, the matrix γ_5 anticommutes with the other γ's and it is "normalized" as $\gamma_5^2 = 1$. Both operators (2.8) are Hermitean with respect to the standard inner product $(\varphi, \varphi') = \int \varphi^+\varphi'\sqrt{g}d^4x$. According to Eq. (1.2), the canonical

[3]We define the spinor derivative as $\nabla_\mu = \partial_\mu + \Gamma_\mu$, where $\Gamma_\mu = \frac{1}{8}[\gamma^\lambda, \gamma^\rho]\ V_\rho^i \nabla_\mu V_{i\lambda}$ is the connection and V_ν^i are the tetrades.

[4] It is easy to check that H_s^2 and H_d are Hermitean operators with respect to the following inner products $(\phi', \phi) = \int_{\mathcal{B}} \sqrt{{}^{(3)}g|g_{00}|^{-1}}\ d^3x\ (\phi')^*\phi$, $(\psi', \psi) = \int_{\mathcal{B}} \sqrt{{}^{(3)}g}\ d^3x\ (\psi')^+\psi$ where ${}^{(3)}g = \det g_{ab}$ and $(\psi')^+$ denotes Hermitean conjugation.

free energy vanishes at zero temperature. It is convenient to define the Euclidean free energy so that it have the same property, i.e., as

$$F_i^E[g,\beta] = \beta^{-1}W_i[g,\beta] - E_i^0[g] \quad , \tag{2.9}$$

$$E_i^0[g] = \lim_{\beta\to\infty}\left(\beta^{-1}W_i[g,\beta]\right) \quad . \tag{2.10}$$

The quantities $E_i^0[g]$ have the meaning of the vacuum energy. Note that $F_i^E[g,\beta]$ and $E_i^0[g,\beta]$ are covariant functionals of the metric, because the Euclidean actions $W_i[g,\beta]$ are covariant at any values of β.

Our aim is to find the relation between F^E and F^C. The important property of F^C is that it can be represented in the form similar to Eq. (2.9)

$$F_i^C[g,\beta] = \beta^{-1}\bar{W}_i[g,\beta] - \bar{E}_i^0[g] \quad , \tag{2.11}$$

$$\bar{E}_i^0[g] = \lim_{\beta\to\infty}\left(\beta^{-1}\bar{W}_i[g,\beta]\right) \quad . \tag{2.12}$$

The functionals $\bar{W}_i[g,\beta]$ are the following effective actions

$$\bar{W}_s[g,\beta] = \frac{1}{2}\log\det\bar{L}_s \quad , \quad \bar{W}_d[g,\beta] = -\log\det\bar{L}_d \quad . \tag{2.13}$$

The operators $\bar{L}_i$ are related to L_i, Eq. (2.8), by the conformal transformations

$$\bar{L}_s = e^{-3\sigma}L_s\, e^{\sigma} \quad , \quad \bar{L}_d = e^{-\frac{5}{2}\sigma}L_d\, e^{\frac{3}{2}\sigma} \quad , \tag{2.14}$$

where $e^{-2\sigma} = g_{\tau\tau}$. Representation (2.11) is well known [6]-[11] but for the sake of completeness we give its derivation in Appendix A. The key point is that $\bar{L}_i$ are expressed in terms of the one-particle Hamiltonians

$$\bar{L}_s = \bar{H}_s^2 - \partial_\tau^2 \quad , \tag{2.15}$$

$$\bar{L}_d = \gamma_5\bar{\gamma}_\tau(\bar{H}_d + \partial_\tau) \quad , \quad \bar{L}_d^2 = \bar{H}_d^2 - \partial_\tau^2 \quad , \tag{2.16}$$

$$\bar{H}_s^2 = e^{-\sigma}H_s^2 e^{\sigma} \quad , \quad \bar{H}_d = e^{-\frac{3}{2}\sigma}\, H_d\, e^{\frac{3}{2}\sigma} \quad , \tag{2.17}$$

where $\bar{\gamma}_\tau = e^{\sigma}\gamma_\tau$. Note that according to Eqs. (2.17) the spectra of the operators $\bar{H}_i$ and H_i, Eqs. (2.4), (2.5), coincide, which means that $\bar{H}_i$ are simply an another representation of H_i. The operators $\bar{L}_i$ act on the fields on the ultrastatic space $\bar{\mathcal{M}}_\beta$ with the metric

$$d\bar{s}^2 = d\tau^2 + \bar{g}_{ab}dx^a dx^b \quad , \quad 0 \le \tau \le \beta \tag{2.18}$$

which is conformally related to metric (2.6), $\bar{g}_{\mu\nu} = g_{\mu\nu}/|g_{00}|$. One can show that

$$\bar{H}_d = i\bar{\gamma}_0(\bar{\gamma}^a\bar{\nabla}_a + e^{-\sigma}m) \quad , \tag{2.19}$$

where $\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2\bar{g}^{\mu\nu}$, and

$$\bar{H}_i^2 = -\bar{\nabla}^a\bar{\nabla}_a + e^{-2\sigma}m^2 + V_i \quad . \tag{2.20}$$

The derivatives $\bar{\nabla}_\alpha$ are defined with the help of the metric $\bar{g}_{ab}$ of the surface $\tau = const$ in ultrastatic space (2.18). This surface is conformally related to $\mathcal{B}$ and we denote it $\bar{\mathcal{B}}$. The "potential terms" V_i are

$$V_s = \xi \bar{R} + e^{-2\sigma}(1-6\xi)\left(\nabla^\mu w_\mu - w^\mu w_\mu\right) \quad , \tag{2.21}$$

$$V_d = \frac{1}{4}\bar{R} + e^{-2\sigma} m \gamma^\mu w_\mu \quad . \tag{2.22}$$

Here $w_\mu = (0, w_a)$ is the four vector of acceleration and $\bar{R}$ is the curvature of the ultrastatic background

$$\bar{R} = e^{-2\sigma}\left(R + 6(\nabla^\mu w_\mu - w^\mu w_\mu)\right) \quad . \tag{2.23}$$

The formulation of statistical mechanics in terms of the theory on ultrastatic spaces was developed by Dowker and Kennedy [9] and by Dowker and Schofield [13],[14]. Relations (2.19)-(2.23) coincide with those used in Refs.[13],[14].

The actions W_i and $\bar{W}_i$ are determined by the conformally related wave operators, see Eqs. (2.14). In static space-times without horizons the renormalized functionals W_i and $\bar{W}_i$ differ by the anomalous terms computed in [13],[14]. These terms, however, are proportional to β, so they result in the difference between vacuum energies (2.10) and (2.12). The Euclidean and the canonical free energies in this case coincide. When there is a horizon the conformal transformation to the ultrastatic metric becomes singular and this case requires a special analysis.

3 Covariant Euclidean formulation: conical singularities and divergences

To begin with we describe the class of space-times with Killing horizons which will be discussed here. We assume that the metric (2.2) near the bifurcation surface Σ has the following form

$$ds^2 = g_{00}(\theta,\rho)dt^2 + d\rho^2 + \gamma_{pq}(\theta,\rho)d\theta^p d\theta^q \quad , \quad p,q = 1,2 \quad . \tag{3.1}$$

In this representation the location of Σ is determined by the equation $\rho = 0$, θ^p are the coordinates on this surface. We also assume that near $\rho = 0$ the components of the metric are decomposed as

$$g_{00}(\theta,\rho) = -\kappa^2\rho^2\left(1 - \frac{1}{6}\rho^2 R_{ijij}(\theta) + O(\rho^4)\right) \quad , \tag{3.2}$$

$$\gamma_{pq}(\theta,\rho) = \tilde{\gamma}_{pq}(\theta) + \rho^2 h_{pq}(\theta) + O(\rho^4) \quad , \tag{3.3}$$

where $\tilde{\gamma}_{pq}$ is the metric tensor on Σ. The constant κ is called the surface gravity. Metrics which obey the properties (3.1)-(3.3) correspond to static nonextremal black holes. It

can be shown that the quantities R_{ijij}, R_{ii} are the projections of the Riemann and Ricci tensors calculated on Σ on the directions normal to this surface. Namely,

$$R_{ii} = R_{\mu\nu} n_i^\mu n_i^\nu \quad , \quad R_{ijij} = R_{\mu\lambda\nu\rho} n_i^\mu n_j^\lambda n_i^\nu n_j^\rho \quad . \tag{3.4}$$

Here n_i^μ are two unit orthonormal vectors orthogonal to Σ and the summation over the indexes $i, j = 1, 2$ is assumed. It can be also shown that

$$h_{pq}\tilde{\gamma}^{pq} = \frac{1}{2}(R_{ijij} - R_{ii}) \quad . \tag{3.5}$$

Near Σ the Euclidean section $\mathcal{M}_\beta$ of space-time (3.1) looks as

$$ds^2 \simeq \kappa^2\rho^2 dt^2 + d\rho^2 + \tilde{\gamma}_{pq} d\theta^p d\theta^q \quad , \quad 0 \leq \tau \leq \beta \quad . \tag{3.6}$$

This space is regular when $\beta^{-1} = \beta_H^{-1} = \frac{\kappa}{2\pi}$, where the constant β_H^{-1} is called the Hawking temperature. For arbitrary β there are conical singularities and $\mathcal{M}_\beta$ looks as $\mathcal{C}_\beta \times \Sigma$, where $\mathcal{C}_\beta$ is a cone.

As a result of conical singularities, the Euclidean free energy F^E is divergent even after subtracting from it the vacuum energy. The divergent part F^E_{div} of F^E can be calculated with the help of different regularizations. We begin with the dimensional regularization and consider D-dimensional space-time. It is assumed that when going to arbitrary dimensions the background space holds its Killing structure and equations (3.1)-(3.3) do not change. The difference now is that the tensors $R_{\mu\nu\lambda\rho}$, $R_{\mu\nu}$ are calculated in D dimensions and Σ is a $(D-2)$ dimensional surface. The divergent part of the Euclidean free energy is

$$F^E_{\mathrm{div}}[g,\beta,D] = -\eta \frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}} \frac{\pi^2 m^{D-4}}{3\kappa\beta^2} \int_\Sigma \left[f_1 m^2 - \left(p_1 \frac{4\pi^2}{\kappa^2\beta^2} P + p_2 R + p_3 R_{ii} \right) \right] \quad , \tag{3.7}$$

where D is considered as a complex parameter and the integral is taken over the bifurcation surface Σ, $\int_\Sigma \equiv \int_\Sigma \sqrt{\tilde{\gamma}} d^{D-2}\theta$. We put $P = 2R_{ijij} - R_{ii}$ and introduce the constants f_1 and p_k which are listed in Table 1:

spin	f_1	p_1	p_2	p_3
0	1	$\frac{1}{60}$	$\frac{1}{6} - \xi$	0
$\frac{1}{2}$	$-\frac{1}{2} r_d$	$-\frac{7}{480} r_d$	$\frac{1}{24} r_d$	$-\frac{1}{16} r_d$

Table 1.

Here r_d is the dimensionality of the spinor representation, $r_d = 4$ for Dirac spinors and $r_d = 2$ for massless Weyl spinors. As follows from Eq. (3.7), in the dimensional regularization F^E has a simple pole at $D = 4$. The dimensional regularization reproduces the divergences

of the logarithmical type only. For this reason it is also worth studying $F^E_{\rm div}$ in Pauli-Villars regularization which usually gives all divergent terms. The Pauli-Villars method is based on introduction of several, say, 5 additional fields. 2 fields with masses M_k have the same statistics as the original field, while other 3 ones with masses M'_r have the wrong statistics, i.e., they are fermions for scalars and bosons for spinors. The latter fields give contribution to F^E with the sign opposite to that of physical fields. To eliminate the divergences two restrictions are imposed

$$m^p + \sum_k M_k^p - \sum_r (M'_r)^p = 0 \quad , \quad p = 2, 4 \quad . \tag{3.8}$$

These equations can be resolved by choosing [22] $M_{1,2} = \sqrt{3\mu^2 + m^2}$, $M'_{1,2} = \sqrt{\mu^2 + m^2}$, $M'_3 = \sqrt{4\mu^2 + m^2}$. The divergences in Pauli-Villars regularization can be obtained from Eq. (3.7). By adding contributions of the regulator fields with the sign corresponding to their statistics and taking the limit $D \to 4$, which is finite due to restriction (3.8) with $p = 2$, one finds

$$F^E_{\rm div}[g, \beta, \mu] = -\frac{\eta}{48\kappa\beta^2} \int_\Sigma \left[b f_1 + a \left(p_1 \frac{4\pi^2}{\kappa^2\beta^2} P + p_2 R + p_3 R_{ii} \right) \right] \quad , \tag{3.9}$$

$$a = a(m, \mu) = -\ln m^2 - \sum_k \ln M_k^2 + \sum_r \ln (M'_r)^2 \quad , \tag{3.10}$$

$$b = b(m, \mu) = m^2 \ln m^2 + \sum_k M_k^2 \ln M_k^2 - \sum_r (M'_r)^2 \ln (M'_r)^2 \quad . \tag{3.11}$$

The parameter μ^2 plays the role of the ultraviolet cutoff. The regularization is removed when $\mu \to \infty$. In this limit $a \sim \ln(\mu^2/m^2)$ and $b \sim \mu^2$ $(a, b > 0)$. Thus, in general, $F^E_{\rm div}$ includes both logarithmic and quadratic divergences.

The derivation of Eq. (3.7) is standard. In the dimensional regularization the Schwinger-DeWitt proper-time representation [23] gives the divergent part of the effective action in the form

$$W^E_{\rm div}[g, \beta] = -\frac{\eta}{2} \int_0^\infty \frac{ds}{s} e^{-m^2 s} \frac{1}{(4\pi s)^{D/2}} \left(B_0 + s B_1 + s^2 B_2 \right) \quad . \tag{3.12}$$

Here B_k are the Hadamard-Minackshisundaram-DeWitt-Seeley (or heat) coefficients of the heat kernel asymptotic expansion[5]

$$\mathrm{Tr}\ e^{-s\triangle} \approx \frac{1}{(4\pi s)^{D/2}} \left(B_0 + s B_1 + s^2 B_2 + ... \right) \quad . \tag{3.13}$$

The Laplacians $\triangle$ look as $\triangle = -\nabla^\mu \nabla_\mu + X$, where ∇_μ is the covariant derivative defined according with the spin; $X = (1/6 - \xi) R$ for scalars, and $X = \frac{1}{4} R$ for spinors. The relation between $\triangle$'s and the operators L_s and L_d, Eqs. (2.8), is

$$L_s = \triangle_s + m^2 \quad , \quad L_d^2 = \triangle_d + m^2 \quad . \tag{3.14}$$

[5]We do not take into account the boundaries of space-time.

On singular spaces $\mathcal{M}_\beta$ the coefficients B_k can be represented as the sum of two terms

$$B_k = A_k + A_{\beta,k} \quad . \tag{3.15}$$

A_k have the form of the standard coefficients defined on the regular domain of $\mathcal{M}_\beta$, $A_{\beta,k}$, $k \geq 1$, are functionals on Σ which appear because of conical singularities. For scalars and spinors the first two coefficients have the form

$$A_{\beta,1} = f_1 \frac{\pi}{3\gamma}(\gamma^2 - 1)\mathcal{A} \quad , \tag{3.16}$$

$$A_{\beta,2} = \frac{\pi}{3\gamma} \int_\Sigma \left[(\gamma^4 - 1)p_1 P + (\gamma^2 - 1)(p_2 R + p_3 R_{ii})\right] \quad , \tag{3.17}$$

where $\gamma = \frac{\beta_H}{\beta}$ and the numbers f_1 and p_k are given in Table 1. Expression (3.9) for $F^E_{\mbox{div}}[g,\beta]$ follows from Eqs. (2.9), (2.10) and (3.12),(3.15)-(3.17).

For scalars the coefficient $A_{\beta,1}$ was found by Cheeger [24], see also Refs. [17],[15]. The spinor coefficient $A_{\beta,1}$ follows from the results of Refs. [25],[26]. The scalar coefficient $A_{\beta,2}$ and the general structure of the higher coefficients $A_{\beta,k}$ were analysed in Refs. [16],[27],[28]. The calculation of the spinor coefficient $A_{\beta,2}$ is our new result. Its derivation is similar to that of Ref. [16] but has new features related to the spin. The reader can find the details in Appendix B.

4 Canonical formulation: infrared divergences

Our aim now is to investigate the divergences of the canonical free energy and compare them with the results (3.7),(3.9) found in the Euclidean formulation. We begin with remarks concerning specific features of quantum systems in the presence of horizons. In this case the one-particle oscillators of fields have a continuous spectrum of frequencies $0 \leq \omega < \infty$. The eigenvalues of the operators $\bar{H}_i$, Eqs. (2.19),(2.20), run down to $\omega = 0$ even for massive fields, i.e., the usual mass gap is absent. This property has the simple explanation. $\bar{H}_i$ are given on the space $\bar{\mathcal{B}}$, which is the spatial part of ultrastatic space (2.18) related to the original space (2.6) by the conformal transformation. On the ultrastatic space the location of the horizon is mapped at infinity and $\bar{\mathcal{B}}$ turns out to be non-compact. At the same time, the masses m_i of the fields can be neglected near the horizon because they are multiplied by the vanishing factor $e^{-2\sigma} = |g_{00}|$. Regarding the "potential" terms in $\bar{H}_i$, see Eqs. (2.21),(2.22), they are constant and negative on the horizon, $V_s = -\kappa^2$, $V_d = -\frac{3}{2}\kappa^2$, and look as tachionic masses [20],[21]. As a result, the densities $\frac{dn_i}{d\omega}$ of the eigen-values of $\bar{H}_i$ blow up near the horizon in an infrared way.

To investigate this divergence the following method can be used. By the definition, the trace of the heat kernel of the operator $\bar{H}_i^2$ is

$$\mbox{Tr}\ e^{-\bar{H}_i^2 t} = \int_0^\infty d\omega \frac{dn_i(\omega)}{d\omega} e^{-\omega^2 t} \quad . \tag{4.1}$$

By making use of the inverse Laplace transform one obtains

$$\frac{dn_i(\omega)}{d\omega} = \frac{\omega}{\pi}\int_{-\infty}^{+\infty} d\alpha \; e^{i\alpha\omega^2} \mathrm{Tr}\; e^{-i\alpha \bar{H}_i^2} \quad . \tag{4.2}$$

The diagonal matrix elements $\langle x|\exp(-\bar{H}_i^2 t)|x\rangle \equiv \left[\exp(-\bar{H}_i^2 t)\right]_{\mathrm{diag}}$ are well defined but the traces involve the integration over the non-compact space $\bar{\mathcal{B}}$ and diverge. The key observation is that the divergent parts of $\frac{dn_i(\omega)}{d\omega}$ can be found by making use of the asymptotic properties of the traces at small t, which are very well known.

Let us begin with massless fields in the Rindler space

$$ds^2 = -\kappa^2\rho^2 dt^2 + d\rho^2 + dx^2 + dy^2 \quad , \quad -\infty < x, y < \infty \quad , \quad \rho > 0 \quad . \tag{4.3}$$

The Rindler horizon is the plane $\mathrm{I\!R}^2$, ρ is the proper distance to the horizon. The Rindler space can be considered as an approximation to the black hole geometry near the horizon. The metric on the corresponding space $\bar{\mathcal{B}}$ is

$$dl^2 = \kappa^{-2}\rho^{-2}(d\rho^2 + dx^2 + dy^2) \quad . \tag{4.4}$$

As can be shown, $\bar{\mathcal{B}}$ coincides with the hyperbolic manifold $\mathrm{I\!H}^3$ having the constant curvature $\bar{R} = -6\kappa^2$ [29]. A review of the heat kernels of Laplace operators on such spaces can be found in Refs. [29],[30]. It is remarkable that for the massless fields the heat kernels are known explicitly and their diagonal elements read[6]

$$\left[e^{-t\bar{H}_s^2}\right]_{\mathrm{diag}} = \frac{1}{(4\pi t)^{3/2}} \quad , \quad \left[e^{-t\bar{H}_d^2}\right]_{\mathrm{diag}} = \frac{r_d}{(4\pi t)^{3/2}}\left[1 + \frac{1}{2}\kappa^2 t\right] \quad . \tag{4.5}$$

The number r_d appears in the spinor case after tracing over the spinor indexes. Obviously, the traces of these operators diverge at $\rho = 0$. Let us restrict the integration in the traces by values $\rho \geq \epsilon$ where ϵ is a proper distance to the horizon. Such a method is called the volume cutoff. With the help of Laplace transform (4.2) (see Ref. [31]) one easily obtains from (4.5) the divergences of the densities of levels at $\epsilon \to 0$

$$\left[\frac{dn_s(\omega,\epsilon)}{d\omega}\right]_{\mathrm{div}} = \frac{\mathcal{A}}{4\pi^2\kappa^3}\frac{\omega^2}{\epsilon^2} \quad , \quad \left[\frac{dn_d(\omega,\epsilon)}{d\omega}\right]_{\mathrm{div}} = r_d \frac{\mathcal{A}}{4\pi^2\kappa^3}\left[\frac{\omega^2}{\epsilon^2} + \frac{\kappa^2}{4\epsilon^2}\right] \tag{4.6}$$

where $\mathcal{A}$ formally stands for the area of the horizon [7]. For scalar fields our result agrees with previous computations by the WKB method, see, for instance Ref. [19]. A generalization of (4.6) to massive scalars was explicitly found in Refs. [20],[21].

Let us consider how do Eqs. (4.6) modify when the geometry deviates from the Rindler form. The spaces $\bar{\mathcal{B}}$ can be approximated by $\mathrm{I\!H}^3$ only in the limit $\rho \to 0$. For this reason, the diagonal elements of the heat kernels of $\bar{H}_i^2$ on $\bar{\mathcal{B}}$ are represented by the Taylor series

[6]The expression for the scalar kernel is given in Ref. [29], the spinor kernel follows from the ζ function which is also given there.

[7]Since the Rindler horizon is a plane, only the densities of levels per unit area have a strict meaning.

in ρ^2 converging at $\rho \to 0$ to expressions (4.5). Presumably, the coefficients in these series should be the local functions of the curvature and so to find them it is sufficient to use the asymptotic form of the traces. One has

$$\left[e^{-\bar{H}_i^2 t}\right]_{\text{diag}} \simeq \frac{1}{(4\pi t)^{3/2}}\left(r_i + \bar{a}_{i,1}t + \bar{a}_{i,2}t^2 + ...\right) \quad , \tag{4.7}$$

where $r_s = 1$ and $\bar{a}_{i,n}$ are the diagonal elements of the corresponding heat coefficients. As before the summation over the spinor indexes in (4.7) is assumed. With the help of Eqs. (2.21)-(2.23) one finds

$$\bar{a}_{s,1} = e^{-2\sigma}\left[\left(\frac{1}{6} - \xi\right)R - m^2\right] \quad , \tag{4.8}$$

$$\bar{a}_{d,1} = -e^{-2\sigma} r_d \left[\frac{1}{12}R + \frac{1}{2}(\nabla^\mu w_\mu - w^\mu w_\mu) + m^2\right] \quad . \tag{4.9}$$

These expressions can be decomposed in powers of ρ^2 by making use of Eqs. (3.1)-(3.5)

$$\nabla^\mu w_\mu = -R_t^t = -\frac{1}{2}R_{ii} + O(\rho^2) \quad , \tag{4.10}$$

$$w^\mu w_\mu = \frac{1}{\rho^2}\left(1 - \frac{1}{3}\rho^2 R_{ijij} + O(\rho^4)\right) \quad , \tag{4.11}$$

$$\bar{a}_{s,1} = \kappa^2\rho^2\left[\left(\frac{1}{6} - \xi\right)R - m^2\right] + O(\rho^4) \quad , \tag{4.12}$$

$$\bar{a}_{d,1} = r_d\frac{\kappa^2}{2}\left[1 + \rho^2\left(\frac{1}{2}R_{ii} - \frac{1}{2}R_{ijij} - \frac{1}{6}R - 2m^2\right) + O(\rho^4)\right] \quad , \tag{4.13}$$

where R is evaluated at $\rho = 0$. For the massless fields in the Rindler space $\bar{a}_{s,1} = 0$, while $\bar{a}_{d,1} = r_d\frac{1}{2}\kappa^2$, in agreement with Eqs. (4.5). Decompositions (4.12), (4.13) are written explicitly up to the terms of the order ρ^2, other terms do not contribute to the divergence of the traces.

Analogous decompositions can be found for the heat coefficients $\bar{a}_{i,n}$ with $n \geq 2$, however, $\bar{a}_{i,n}$ vanish at small ρ faster than ρ^2. To see this let us consider the operators $\bar{L}_s$, $\bar{L}_d^2$ which are given on the ultrastatic space $\bar{\mathcal{M}}_\beta$, Eq. (2.18). $\bar{L}_s$, $\bar{L}_d^2$ are related to the three dimensional Hamiltonians $\bar{H}_i^2$ by Eqs. (2.15),(2.16). Because $\bar{\mathcal{M}}_\beta = S^1 \times \bar{\mathcal{B}}$ the diagonal parts of the heat coefficients of $\bar{L}_s$, $\bar{L}_d^2$ coincide with $\bar{a}_{i,n}$. On the other hand, $\bar{L}_s$, $\bar{L}_d^2$ are related by conformal transformation (2.14) to the covariant Euclidean operators L_s, L_d^2. In this case, according to Dowker and Schofield [13],[14],

$$\bar{a}_{i,2} = e^{-4\sigma}\left(a_{i,2} + \nabla_\mu J_i^\mu\right) \quad , \tag{4.14}$$

where $a_{i,2}$ are the heat coefficients corresponding to L_s, L_d^2. The currents J_i^μ have the form[8]

$$J_s^\mu = -\frac{1}{45}\left\{5\left[\frac{1}{2}\nabla^\mu(w^2) + w^\mu(w^2) - w^\mu\nabla w\right] - \frac{3}{2}\nabla^\mu\nabla w - 2R^{\mu\nu}w_\nu\right.$$

[8]Note that our signature of the metric is different from that in Refs. [13],[14].

$$+\frac{3}{2}w^\mu R + 15\left[\left(\xi-\frac{1}{6}\right)R + m^2\right]w^\mu\Big\} \quad , \tag{4.15}$$

$$J_d^\mu = -\frac{r_d}{45}\Big\{-14\left[\frac{1}{2}\nabla^\mu(w^2) + w^\mu(w^2) - w^\mu\nabla w\right] + 18\nabla^\mu\nabla w + 22R^{\mu\nu}w_\nu$$
$$-(5R+30m^2)w^\mu\Big\} \quad , \tag{4.16}$$

where $w^2 = w^\nu w_\nu$ and $\nabla w = \nabla^\nu w_\nu$. The coefficients $a_{i,2}$ are regular at $\rho \to 0$. With the help of Eqs. (4.10),(4.11) one can check that $\nabla_\mu J_i^\mu$ are regular as well. Therefore, as follows from (4.14), $\bar{a}_{i,2} \sim e^{-4\sigma} \sim \rho^4$. The other coefficients $\bar{a}_{i,n}$ with $n > 2$ are determined from $\bar{a}_{i,2}$ by recursion relations [23] and so $\bar{a}_{i,n}$ vanish near Σ at least as fast as $\bar{a}_{i,2}$.

Our conclusion is that only $\bar{a}_{i,1}$ contribute to the divergences of the traces. From Eqs. (4.2),(4.7),(4.12),(4.13) we obtain the divergent part of the densities of levels

$$\left[\frac{dn_s(\omega,\epsilon)}{d\omega}\right]_{\text{div}} = \frac{1}{4\pi^2\kappa^3}\int_\Sigma \left\{\omega^2\left(\frac{1}{\epsilon^2} - \frac{1}{4}P\ln\frac{\epsilon^2}{l^2}\right) - \frac{\kappa^2}{2}\ln\frac{\epsilon^2}{l^2}\left[\left(\frac{1}{6}-\xi\right)R - m^2\right]\right\} \quad , \tag{4.17}$$

$$\left[\frac{dn_d(\omega,\epsilon)}{d\omega}\right]_{\text{div}} = \frac{r_d}{4\pi^2\kappa^3}\int_\Sigma \Big\{\omega^2\left(\frac{1}{\epsilon^2} - \frac{1}{4}P\ln\frac{\epsilon^2}{l^2}\right) + \frac{\kappa^2}{4\epsilon^2}$$
$$-\frac{\kappa^2}{2}\ln\frac{\epsilon^2}{l^2}\left(\frac{1}{8}R_{ii} - \frac{1}{12}R - m^2\right)\Big\} \quad . \tag{4.18}$$

Here P is the quantity defined after Eq. (3.7) and l is an additional infrared cutoff parameter imposed at a large distance from the horizon. For massive fields $l \simeq m^{-1}$.

The following remarks concerning expressions (4.17) and (4.18) are in order. As compared to the computations on the Rindler space, Eqs. (4.6), there are logarithmic corrections from the mass of the fields and non-zero curvature near the horizon. Because of the logarithmic terms the densities $\frac{dn_i}{d\omega}$ have a different behavior at small frequencies for scalar and spinor fields. Spinor density (4.18) is positive at small ϵ in the whole range of frequencies. Contrary to this, scalar density (4.17) at $\omega \to 0$ is proportional to $\kappa^2 m^2 \ln\epsilon^2/l^2$ and is negative (note that curvature corrections are negligible with respect to the mass of the field). Such a feature indicates that for scalars the description of the modes with low frequencies, the so called soft modes, may need a modification Ref. [32].

From Eqs. (1.2),(4.17),(4.18) we get the divergent parts of the canonical free energies

$$F^C_{s,\text{div}}[g,\beta,\epsilon] = -\frac{1}{\kappa^3}\int_\Sigma \left\{\frac{\pi^2}{180\beta^4\epsilon^2} - \left[\frac{\pi^2}{720\beta^4}P + \frac{\kappa^2}{48\beta^2}\left(\left(\frac{1}{6}-\xi\right)R - m^2\right)\right]\ln\frac{\epsilon^2}{l^2}\right\} \quad , \tag{4.19}$$

$$F^C_{d,\text{div}}[g,\beta,\epsilon] = -\frac{r_d}{\kappa^3}\int_\Sigma \Big\{\left(\frac{7\pi^2}{1440\beta^4} + \frac{\kappa^2}{192\beta^2}\right)\frac{1}{\epsilon^2}$$
$$-\left[\frac{7\pi^2}{5760\beta^4}P + \frac{\kappa^2}{96\beta^2}\left(\frac{1}{8}R_{ii} - \frac{1}{12}R - m^2\right)\right]\ln\frac{\epsilon^2}{l^2}\Big\} \quad . \tag{4.20}$$

These equations enable one to calculate the divergences of the other characteristics of canonical ensembles. In particular, one can see that the statistical-mechanical entropy

diverges near the Killing horizon and in the leading asymptotic it is proportional to the area $\mathcal{A}$ of the horizon. For scalar fields this leading asymptotic coincides with the WKB results by t'Hooft [19] and many other authors. Equations (4.19) and (4.20) also follow from the high-temperature expansions obtained by Dowker and Schofield [13],[14]. Application of these results to our case is justified because when approaching the horizon the local temperature unlimitedly grows.

5 Canonical formulation: ultraviolet divergences

Comparison of Eq. (3.9) with Eqs. (4.19),(4.20) shows that the divergences of the Euclidean and canonical free energies are expressed in terms of the similar geometrical quantities but have different dependence on the temperature. Also the nature of the divergences is different: F^E diverges in an ultraviolet way while the ϵ-divergence in F^C has an infrared origin. Finally, there is one more important difference between the regularizations of F^E and F^C. The ultraviolet regularizations are usually applied to operators and functionals but not to the background field itself. Contrary to this, the volume cutoff regularization makes the space incomplete and modifies the background field essentially.

In the presence of the horizon the densities of levels have a remarkable property. Namely, there are regularizations of $\frac{dn}{d\omega}$ which make it possible to remove the infrared cutoff near the horizon and to define the densities on the complete background. As a result, $\frac{dn}{d\omega}$ acquire new divergences which correspond exactly to the ultraviolet divergences of the covariant Euclidean theory.

As the first example, let us consider the dimensional regularization. The power of the leading divergency in Eqs. (4.17),(4.18) is determined by the dimensionality of the space $\bar{\mathcal{B}}$. In D-dimensional space-time the leading divergence is ϵ^{2-D}, if $D \neq 2$, and at $D > 2$ one can take the limit $\epsilon \to 0$. After the analytical continuation to the complex values of D the quantities $\frac{dn}{d\omega}$, have a pole at $D = 4$. The method how to investigate $\frac{dn}{d\omega}$ near the pole is the following. As before we use relation (4.2). By taking into account the form of the operators $\bar{H}_i^2$, see Eq. (2.20), we can write

$$\left[e^{-\bar{H}_i^2 t}\right]_{\text{diag}} \simeq \frac{1}{(4\pi t)^{(D-1)/2}} e^{-m^2\kappa^2\rho^2 t} \left(r_i + \bar{b}_{i,1} t + \bar{b}_{i,2} t^2 + ...\right) \quad , \tag{5.1}$$

$$\bar{b}_{s,1} = \kappa^2\rho^2 \left(\frac{1}{6} - \xi\right) R + O(\rho^4) + O(D-4) \quad , \tag{5.2}$$

$$\bar{b}_{d,1} = r_d \frac{\kappa^2}{2} \left[1 + \rho^2 \left(\frac{1}{2} R_{ii} - \frac{1}{2} R_{ijij} - \frac{1}{6} R\right) + O(\rho^4) + O(D-4)\right] \quad , \tag{5.3}$$

$$\bar{b}_{s,2} = O(\rho^4) + O(D-4,\rho^2) \quad , \quad \bar{b}_{d,2} = \frac{1}{2} r_d \kappa^4 m^2 \rho^2 + O(\rho^4) + O(D-4,\rho^2) \quad . \tag{5.4}$$

The coefficients $\bar{b}_{i,n}$ are found with the help of Eqs. (4.7),(4.12)-(4.14) The terms $O(D-4)$, $O(D-4,\rho^2)$ denote additions which appear in formulas (4.12)-(4.14) when $D \neq 4$. We do

not write these terms explicitly because they do not result in singularities when $D \to 4$. The integration measure in the traces is obtained from Eqs. (3.1)-(3.5)

$$\int_{\bar{B}} \sqrt{\bar{g}} d^{D-1}x \simeq \frac{1}{\kappa^{D-1}} \int_{\Sigma} \int_0^{\infty} \rho^{1-D} d\rho \left[1 + \frac{1}{4}\rho^2 P + O(D-4, \rho^2)\right] \quad . \tag{5.5}$$

After substitution of Eqs. (5.2)-(5.4) in Eq. (5.1) and making use of (5.5) we can derive the singular part of the traces

$$\left[\mathrm{Tr} e^{-\bar{H}_s^2 t}\right]_{\mathrm{div}} = \frac{\Gamma\left(1 - \frac{D}{2}\right)}{(4\pi)^{(D-1)/2}} \frac{m^{D-4}}{2\kappa t^{3/2}} \int_{\Sigma} \left[\left(m^2 - \left(\frac{1}{6} - \xi\right) R\right) t - \frac{P}{4\kappa^2}\right] \quad , \tag{5.6}$$

$$\left[\mathrm{Tr} e^{-\bar{H}_d^2 t}\right]_{\mathrm{div}} = r_d \frac{\Gamma\left(1 - \frac{D}{2}\right)}{(4\pi)^{(D-1)/2}} \frac{m^{D-4}}{2\kappa t^{3/2}} \int_{\Sigma} \left[\left(m^2 + \frac{R}{12} - \frac{R_{ii}}{8}\right) t - \frac{P}{4\kappa^2}\right] \quad . \tag{5.7}$$

The Γ-functions appear as a result of the integration over ρ. It should be noted that for the spinors the contribution of the coefficient $\bar{b}_{d,2}$ cancels the pole caused by the term $r_d \kappa^2/2$ in $\bar{b}_{d,1}$. The divergent part of density of energy levels is obtained from (5.6),(5.7) with the help of Eq. (4.2)

$$\left[\frac{dn_s(\omega, D)}{d\omega}\right]_{\mathrm{div}} = \frac{\Gamma\left(1 - \frac{D}{2}\right)}{(4\pi)^{D/2}} \frac{m^{D-4}}{\kappa} \int_{\Sigma} \left[2\left(m^2 - \left(\frac{1}{6} - \xi\right) R\right) - \frac{\omega^2}{\kappa^2} P\right] \quad , \tag{5.8}$$

$$\left[\frac{dn_d(\omega, D)}{d\omega}\right]_{\mathrm{div}} = r_d \frac{\Gamma\left(1 - \frac{D}{2}\right)}{(4\pi)^{D/2}} \frac{m^{D-4}}{\kappa} \int_{\Sigma} \left[2\left(m^2 + \frac{R}{12} - \frac{R_{ii}}{8}\right) - \frac{\omega^2}{\kappa^2} P\right] \quad . \tag{5.9}$$

The divergence represents a simple pole at $D = 4$. Finally, from Eqs. (1.2),(5.8),(5.9) one can find the divergent part of the canonical free energy. We do not write this divergence explicitly because it is exactly the same as that of the Euclidean free energy, Eq. (3.7), computed in the dimensional regularization

$$F^C_{\mathrm{div}}[g, \beta, D] = F^E_{\mathrm{div}}[g, \beta, D] \quad . \tag{5.10}$$

This key equality can be also established in the Pauli-Villars regularization. The regularized density of states in this method is the following quantity

$$\frac{dn_i(\omega, \mu)}{d\omega} \equiv \frac{dn_i(\omega)}{d\omega} + \sum_k \frac{dn_i(\omega, M_k)}{d\omega} - \sum_r \frac{dn_i(\omega, M'_r)}{d\omega} \quad . \tag{5.11}$$

Definition (5.11) takes into account that in the Pauli-Villars method each field is replaced by the "multiplet" of fields. The quantities $\frac{dn_i(\omega, M_k)}{d\omega}$, $\frac{dn_i(\omega, M'_r)}{d\omega}$ are the densities of levels of the Pauli-Villars partners and the fields with the wrong statistics give negative contributions. The number of such fields equals the number of the fields with the correct statistics and the leading ϵ-divergences in Eqs. (4.17),(4.18) are cancelled. Regarding logarithmical divergences $\ln \epsilon^2$, they disappear because of constraint (3.8) with $p = 2$. As a result, regularized densities (5.11) are left finite when $\epsilon \to 0$ and can be defined on the

complete space. When the Pauli-Villars cutoff is removed ($\mu \to \infty$), $\frac{dn_i(\omega,\mu)}{d\omega}$ diverge in an ultraviolet way. The divergences can be inferred from Eqs. (5.8),(5.9) by taking into account constraints (3.8)

$$\left[\frac{dn_s(\omega,\mu)}{d\omega}\right]_{\text{div}} = \frac{1}{(4\pi)^2\kappa}\int_\Sigma \left[2b + a\left(\frac{\omega^2}{\kappa^2}P + 2\left(\frac{1}{6}-\xi\right)R\right)\right] \quad , \tag{5.12}$$

$$\left[\frac{dn_d(\omega,\mu)}{d\omega}\right]_{\text{div}} = r_d\frac{1}{(4\pi)^2\kappa}\int_\Sigma \left[2b + a\left(\frac{\omega^2}{\kappa^2}P - \frac{R}{6} + \frac{R_{ii}}{4}\right)\right] \quad . \tag{5.13}$$

Constants a and b are defined by Eqs. (3.10),(3.11) and diverge in the limit of infinite μ. With the help of Eqs. (5.12),(5.13) one can show that

$$F^C_{\text{div}}[g,\beta,\mu] = F^E_{\text{div}}[g,\beta,\mu] \quad , \tag{5.14}$$

where $F^E_{\text{div}}[g,\beta,\mu]$ is given by Eq. (3.9). It should be noted in conclusion that Pauli-Villars regularization of the canonical free energy was first suggested by Demers, Lafrance and Myers [22] who considered a scalar field on the Reissner-Nordström black hole background. The authors used the WKB method. Although our method is different the results for $F^C_{\text{div}}[g,\beta,\mu]$ in this particular case coincide.

6 Discussion

We are interested in finding the relation between the covariant Euclidean and the canonical formulations of statistical mechanics on curved backgrounds with horizons. Let us discuss first why these formulations are equivalent for spaces without horizons. As we showed in Section 2, the canonical formulation is equivalent to the Euclidean theory on ultrastatic background, $\bar{\mathcal{M}}_\beta$, Eq. (2.18), conformally related to the original space-time $\mathcal{M}_\beta$, Eq. (2.6). The Euclidean actions W_i are determined by the operators L_i on $\mathcal{M}_\beta$, see Eq. (2.8). Analogously, in canonical theory the functionals $\bar{W}_i$ are determined by operators $\bar{L}_i$ on $\bar{\mathcal{M}}_\beta$. The classical actions corresponding to these two types of operators are

$$I_i[g,\varphi_i] = \int_{\mathcal{M}_\beta}\varphi_i^+ L_i\varphi_i\sqrt{g}d^4x \quad , \quad \bar{I}_i[\bar{g},\bar{\varphi}_i] = \int_{\bar{\mathcal{M}}_\beta}\bar{\varphi}_i^+ \bar{L}_i\bar{\varphi}_i\sqrt{\bar{g}}d^4x \quad , \tag{6.1}$$

where the notation φ_i is used for scalars ϕ or spinors ψ. As a result of Eq. (2.14),

$$I_i[g,\varphi_i] = \bar{I}_i[\bar{g},\bar{\varphi}_i] \tag{6.2}$$

for $\bar{\phi} = e^{-\sigma}\phi$ and $\bar{\psi} = e^{-\frac{3}{2}\sigma}\psi$. The transformation from one action to another is not singular and the classical theories on $\mathcal{M}_\beta$ and $\bar{\mathcal{M}}_\beta$ are equivalent. In case of massless spinors and massless scalars with $\xi = \frac{1}{6}$ the operators L_i and $\bar{L}_i$ have the same form, which means that the classical theories are conformally invariant. In general case this invariance does not exist. However, it is still possible to introduce an auxiliary conformal charge in the classical actions and interpret Eq. (6.2) in terms of a pseudo conformal

invariance [13],[14]. According to a common point of view [34],[35] the *bare* quantum actions respect the classical symmetries. Thus, for the bare regularized functionals there is the same equality as for the classical actions[9],

$$W_i[g,\beta]_{\text{bare}} = \bar{W}_i[g,\beta]_{\text{bare}} \quad . \tag{6.3}$$

This relation is not valid for the *renormalized* quantities because the conformal symmetry is broken by the quantum anomalies [34],[35]. The difference between the renormalized actions W_i and $\bar{W}_i$ for scalar and spinor fields was found explicitly by Dowker and Schofield [13],[14]. The anomaly, which is an integral over the Euclidean space, is proportional to β and so it contributes to the vacuum energy only. As a result, the free energies F_i^E and F_i^C, Eqs. (2.9),(2.11), coincide *before* and *after* renormalization.

In case of the horizons $\mathcal{M}_\beta$ and $\bar{\mathcal{M}}_\beta$ have the different topologies, $\mathrm{I\!R}^2 \times \Sigma$ and $S^1 \times \bar{\mathcal{B}}$, respectively. The transformation which relates $\mathcal{M}_\beta$ and $\bar{\mathcal{M}}_\beta$ is singular on the bifurcation surface Σ. So the classical theories are not quite equivalent. On the quantum level the horizons result in the divergences of F_i^E and F_i^C which cannot be eliminated by subtracting the vacuum energy. Moreover, the divergences of F_i^E and F_i^C have the different origins.

Our results suggest a way how can the covariant Euclidean and the canonical formulations be reconciled. We showed that there are regularizations which are applicable to both F_i^E and F_i^C. In such regularizations the canonical free energy F_i^C can be defined on the complete background $\bar{\mathcal{M}}_\beta$ and its divergences are identical to the divergences of F_i^E, see Eqs. (5.10),(5.14). When the functionals F_i^E and F_i^C are well defined, it becomes possible to carry out the transformation from $\mathcal{M}_\beta$ to $\bar{\mathcal{M}}_\beta$ and to interpret it as a conformal symmetry. In analogy with the case without horizons, we can make a hypothesis that (at least for scalars and spinors) the regularized *bare* free energies are identical

$$F_i^E[g,\beta]_{\text{bare}} = F_i^C[g,\beta]_{\text{bare}} \quad . \tag{6.4}$$

It is assumed that the both functionals in (6.4) are considered in the same regularization. To give a strict proof of this equality may be a rather difficult problem. There are examples which enable one its direct check. In Ref. [20] Cognola, Vanzo and Zerbini obtained the free energy of massive scalar fields in the Rindler space in an explicit form, see also Ref. [21]. It can be shown that these results, rewritten in the Pauli-Villars regularization, confirm relation (6.4). Another direct confirmation is possible in two dimensions. Two dimensional massless scalar fields were analysed in Ref. [33] and these results support our hypothesis as well.

Equality (6.4) enables one to apply the methods of quantum field theory to statistical mechanics with the horizons. As a consequence, it justifies the ultraviolet renormalization of statistical-mechanical quantities [36]. Our results are also important for studying the statistical-mechanical foundation of the thermodynamics of black holes. It was realized

[9]It is true if the regularization itself does not break classical symmetries

in the last years that statistical-mechanical computations in this case require an off-shell procedure [37], i.e., considering a black hole at temperatures different from the Hawking value. From this point of view, Eq. (6.4) demonstrates the equivalence of two off-shell methods, the canonical and the conical singularity methods. However, the comparison of the off-shell and on-shell results goes beyond the subject of this paper. As non-minimally coupled scalars show [32], the off-shell and on-shell computations are not always equivalent.

Finally, several remarks about the range of validity of our results are in order. We dealt with static nonextremal black hole backgrounds. The method described in Sections 4, 5 is applicable to the extremal black holes as well, but rotating black holes require an additional analysis. For the extremal black holes the density of levels $\frac{dn}{d\omega}$ has the same property as in nonextremal case. Namely, the Pauli-Villars and dimensional regularizations eliminate the ϵ-divergences. For the scalar fields the corresponding calculations were done in Ref. [22]. Our consideration was also restricted to the scalar and spinor fields. The method to calculate the quantity $\frac{dn}{d\omega}$ can be generalized to the fields of other spins. Finding the correspondence between the canonical and the Euclidean formulations for these fields would be an interesting extension of this work.

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A Canonical free-energy and effective action

Here we give the details how to obtain relation (2.11). We suppose that the system has a discrete spectrum of frequencies. Then the canonical free energy is

$$F_i^C[g,\beta] = \eta_i \beta^{-1} \sum_\omega d_i(\omega) \ln\left(1 - \eta_i e^{-\beta\omega}\right) \quad , \tag{A.1}$$

where $\eta_s = 1$, $\eta_d = -1$ and $d_i(\omega)$ is the degeneracy of the level ω. Equation (1.2) for $F_i^C[g,\beta]$ is obtained in the limit when intervals between the frequencies ω go to zero. The basic identities we use are [38]

$$\ln\left(1 - e^{-\beta\omega}\right) = -\frac{\beta\omega}{2} + \ln\beta\omega + \sum_{k=1}^{\infty} \ln\left(1 + \omega^2 \frac{\beta^2}{4k^2\pi^2}\right) \quad , \tag{A.2}$$

$$\ln\left(1 + e^{-\beta\omega}\right) = -\frac{\beta\omega}{2} + \ln 2 + \sum_{k=0}^{\infty} \ln\left(1 + \omega^2 \frac{\beta^2}{(2k+1)^2\pi^2}\right) \quad . \tag{A.3}$$

Note that

$$\sum_{k=1-q}^{\infty} \ln\left(1 + \omega^2 \frac{\beta^2}{(2k+q)^2\pi^2}\right)$$

$$= -\lim_{z\to 0} \frac{d}{dz}\left[\sum_{k=1-q}^{\infty}\left(\frac{\pi^2(2k+q)^2}{\beta^2} + \omega^2\right)^{-z} - \sum_{k=1-q}^{\infty}\left(\frac{\pi^2(2k+q)^2}{\beta^2}\right)^{-z}\right] \tag{A.4}$$

where $q = 0$ or 1. By using the properties of the Riemann ζ function [38] we find

$$\lim_{z\to 0}\frac{d}{dz}\sum_{k=1}^{\infty}\left(\frac{\pi^2(2k)^2}{\beta^2}\right)^{-z} = -\ln\beta \quad , \quad \lim_{z\to 0}\frac{d}{dz}\sum_{k=0}^{\infty}\left(\frac{\pi^2(2k+1)^2}{\beta^2}\right)^{-z} = -\ln 2 \quad . \tag{A.5}$$

Thus, Eqs. (A.2) and (A.3) are represented in the form

$$\ln\left(1 - \eta_i e^{-\beta\omega}\right) = -\frac{\beta\omega}{2} - \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\zeta_i(z\,|\omega,\beta) \quad , \tag{A.6}$$

$$\zeta_i(z\,|\omega,\beta) = \sum_{k=-\infty}^{\infty}\left[\left(\frac{2\pi}{\beta}(k+l_i)\right)^2 + \omega^2\right]^{-z} \quad , \tag{A.7}$$

where $l_s = 0$ and $l_d = \frac{1}{2}$. The series (A.7) converge at $\mathrm{R}ez > \frac{1}{2}$, so the functions $\zeta_i(z\,|\omega,\beta)$ can be defined at $z \to 0$ with the help of the analytic continuation. By taking into account Eqs. (A.1) and (A.6) we get the canonical free energy in form (2.11)

$$F_i^C[g,\beta] = \beta^{-1}\bar{W}_i[g,\beta] - \bar{E}_i^0[g] \quad , \tag{A.8}$$

$$\bar{W}_i[g,\beta] = \eta_i \frac{1}{2}\sum_\omega d_i(\omega)\lim_{z\to 0}\frac{d}{dz}\zeta_i(z\,|\omega,\beta) \quad , \quad \bar{E}_i^0[g] = \eta_i\sum_\omega d_i(\omega)\frac{\omega}{2} \quad , \tag{A.9}$$

where $\bar{E}_i^0[g]$ is the vacuum energy. The quantities $\bar{W}_i[g,\beta]$ and $\bar{E}_i^0[g]$ diverge at large ω although the free energy itself is finite. To see that $\bar{W}_i[g,\beta]$ is effective action (2.13) one

can regularize this functional with the help of the ζ function method. It is enough to make the replacement

$$\sum_\omega d_i(\omega) \lim_{z\to 0} \frac{d}{dz}\zeta_i(z|\omega,\beta) \to \lim_{z\to 0} \frac{d}{dz}\zeta_i(z|\beta) \tag{A.10}$$

where , according to Eq. (A.7),

$$\zeta_i(z|\beta) = \sum_\omega \sum_{k=-\infty}^{\infty} d_i(\omega) \left[\left(\frac{2\pi}{\beta}(k+l_i)\right)^2 + \omega^2\right]^{-z} \quad . \tag{A.11}$$

$\zeta_i(z|\beta)$ are the generalized ζ functions of the operators $\bar{L}_i$, see Eqs. (2.15),(2.16). For operators on manifolds without boundaries $\zeta_i(z\ |\beta)$ can be defined as a meromorphic function with simple poles at $z = 1, 2$ [30].

B Spinor heat coefficients on conical singularities

Let us consider the heat kernel of the spinor Laplace operator $\triangle_d = -\nabla^\mu\nabla_\mu + \frac{1}{4}R$ on spaces $\mathcal{M}_\beta$ with conical singularities. For simplicity we put $\kappa = 1$, so $\mathcal{M}_\beta$ are regular when $\beta = 2\pi$. The heat kernel obeys the equation

$$(\triangle_d)_x K_\beta(x, x', s) + \partial_s K_\beta(x, x', s) = 0 \quad , \quad K_\beta(x, x', 0) = \delta(x, x') \quad , \tag{B.1}$$

where $\delta(x, x')$ is the delta function on $\mathcal{M}_\beta$. We first describe the heat kernel on a simple cone $\mathcal{C}_\beta$, which will be required for us later. It is known since Sommerfeld that heat kernels on $\mathcal{C}_\beta$ can be expressed in terms of the corresponding heat kernels on the plane $\mathbb{R}^2$. A suitable generalization of the Sommerfeld representation for integer and half-odd-integer spins was given by Dowker [39],[40]. By making use of the results of Ref. [40] we can represent the spinor heat kernel on $\mathcal{C}_\beta$ in the following form

$$K_\beta(x(\tau), x'(0), s) = \frac{1}{2i\beta}\int_A \frac{1}{\sin\frac{\pi}{\beta}(z+\tau)} U(z) K(x(z), x'(0), s) \quad . \tag{B.2}$$

Here τ is the polar-angle coordinate on $\mathcal{C}_\beta$. A is the contour in the complex plane which has two parts. In the upper half-plane it runs from $(\pi - \epsilon) + i\infty$ to $(-\pi + \epsilon) + i\infty$ and in the lower half-plane from $(-\pi + \epsilon) - i\infty$ to $(\pi - \epsilon) - i\infty$. $K(x, x', s)$ is the spinor heat kernel which obeys the problem (B.1) on $\mathbb{R}^2$. The operators $K_\beta(x, x', s)$ and $K(x, x', s)$ correspond to the different spin structures and have the different periodicity. The kernel on $\mathbb{R}^2$ is unchanged when going around the origin of the polar coordinate system. (There is no difference between the origin of the polar coordinates and other points on the plane). Contrary to this, the kernel on $\mathcal{C}_\beta$ changes the sign when τ is increased by β. The covariant derivatives on $\mathbb{R}^2$ are trivial, $\nabla_\mu = \partial_\mu$, but the the covariant derivatives on $\mathcal{C}_\beta$ are defined by the polar tetrades and nontrivial, $\nabla_\mu = \partial_\mu + \Gamma_\mu$. In the basis $\gamma_\mu = (\sigma_1, \sigma_2)$, where σ_k are the Pauli matrices, the spinor connection 1-form is $\Gamma = -\frac{i}{2}\sigma_3 d\tau$. The role of the

matrix U in relation (B.2) is to make a gauge-like transformation from the derivative on $\mathbb{R}^2$ to that on $\mathcal{C}_\beta$,

$$U(\tau)\partial_\mu U^{-1}(\tau) = \nabla_\mu \quad , \quad U(\tau) = \exp\left(\frac{i}{2}\sigma_3\tau\right) \quad . \tag{B.3}$$

In fact, when τ is real $U(\tau)$ is the unitary matrix which is the spinor representation of the rotation on the angle τ.

To find corrections to the heat coefficients from the conical singularities we follow the method suggested in Ref. [16]. According to this method, it is sufficient to work in a narrow domain $\tilde{\Sigma}$ of the singular surface Σ. The rest region of $\mathcal{M}_\beta$ does not have conical singularities and the heat kernel expansion on it has a standard form. $\tilde{\Sigma}$ can be approximated as $\mathcal{C}_\beta \times \Sigma$. So here one can relate $K_\beta(x,x',s)$ with the kernel $K(x,x',s)$ on the regular space $\mathcal{M}_{\beta=2\pi}$ by the formula analogous to Eq. (B.2). The contribution $\mathrm{Tr}_{\tilde{\Sigma}}K_\beta(s)$ from $\tilde{\Sigma}$ to the trace can be written as

$$\mathrm{Tr}_{\tilde{\Sigma}}K_\beta(s) = \mathrm{Tr}_{\tilde{\Sigma}}K(s) + \frac{1}{2i\beta}\int_{A'}\frac{1}{\sin\frac{\pi}{\beta}z}\int_{\tilde{\Sigma}}\sqrt{g}d^Dx\mathrm{Tr}_i\left[U(z)K\left(x(z),x(0),s\right)\right] \quad , \tag{B.4}$$

where Tr_i stands for the trace over the spinor indexes. The points $x(z)$ and $x(0)$ are connected by the integral line of the Killing field ∂_τ. The two terms in r.h.s. of (B.4) appear when contour A is deformed to a small loop around the origin and contour A' which consists of two vertical curves. The effect of conical singularities is related to the second term in (B.4). The asymptotic form of $K(x,x',s)$ is

$$K(x,x',s) \simeq \frac{e^{-\sigma^2(x,x')/4s}}{(4\pi s)^{D/2}}\triangle^{1/2}(x,x')\sum_n a_n(x,x')s^n \quad , \tag{B.5}$$

where $\sigma(x,x')$ is the geodesic distance between points x,x' and $\triangle(x,x')$ is the Van Vleck determinant. The coefficients a_n are determined in terms of the Riemann tensor and its derivatives. Let ρ be the proper distance from the points $x(z)$ and $x(0)$ to Σ. One can find the following relations [16]

$$\sigma^2(x(z),x(0)) \simeq 4\rho^2\sin^2\frac{z}{2} - \frac{1}{6}\rho^4 R_{ijij}\ \sin^2 z \quad , \tag{B.6}$$

$$\triangle^{1/2}(x(z),x(0)) \simeq 1 + \frac{1}{6}\rho^2 R_{ii}\ \sin^2\frac{z}{2} \quad , \tag{B.7}$$

where R_{ii} and R_{ijij} are defined in (3.4). The integration measure on $\tilde{\Sigma}$ can be derived from Eqs. (3.1)-(3.3)

$$\int_{\tilde{\Sigma}}\sqrt{g}d^Dx \simeq \int_\Sigma\sqrt{\gamma}d^{D-2}\theta\int\rho d\rho d\tau\left[1+\rho^2\left(\frac{1}{6}R_{ijij}-\frac{1}{4}R_{ii}\right)\right] \quad . \tag{B.8}$$

For the first coefficients in Eq. (B.5) one finds

$$a_0(x',x) \simeq I + \frac{1}{8}R_{\mu\alpha\lambda\nu}\Sigma^{\nu\lambda}(x')^\mu x^\alpha \quad , \quad a_1(x',x) \simeq -\frac{1}{12}RI \quad , \tag{B.9}$$

where I is the unit matrix in the spinor representation and $\Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. The matrix $U(z)$ corresponds to the rotation of a vector normal to the surface Σ on the angle z. So from Eq. (B.9) we get

$$\mathrm{Tr}_i \left[U(z) a_0 \left(x(z), x(0)\right)\right] \simeq r_d \cos\frac{z}{2} \left(1 + \frac{1}{4}\rho^2 \sin^2\frac{z}{2} R_{ijij}\right) \quad . \tag{B.10}$$

Here r_d is the dimensionality of the spinor representation. Now, it follows from Eqs. (B.5)-(B.7),(B.9) that

$$\mathrm{Tr}_i \left[U(z) K_\beta(x(z), x(0))\right]$$
$$\simeq r_d \frac{\exp\left(-\frac{\rho^2}{s}\sin^2\frac{z}{2}\right)}{(4\pi s)^{D/2}} \cos\frac{z}{2} \left[1 + \rho^2 \left(\frac{1}{4}R_{ijij} + \frac{1}{6}R_{ii}\right)\sin^2\frac{z}{2} + \frac{\rho^4}{24s} R_{ijij} \sin^2 z - \frac{1}{12} Rs\right] \quad . \tag{B.11}$$

It can be shown that approximation (B.11) is sufficient to find all corrections for the first three heat coefficients due to conical singularities. By integrating (B.11) with measure (B.8) we obtain an integral over the surface Σ. Its integrand is an expression linear in the quantities R, R_{ii} and R_{ijij} with coefficients proportional to $\cos\frac{z}{2}\sin^{-2q}\frac{z}{2}$, $q = 1, 2$. Finally, we have to integrate this expression in the complex plane, see. Eq. (B.4). This can be done with the help of formulas [28]

$$\frac{1}{i\beta}\int_{A'} \frac{dz}{\sin\frac{\pi}{\beta}z} \frac{\cos\frac{z}{2}}{\sin^2\frac{z}{2}} = -\frac{1}{3}(\gamma^2 - 1) \quad , \tag{B.12}$$

$$\frac{1}{i\beta}\int_{A'} \frac{dz}{\sin\frac{\pi}{\beta}z} \frac{\cos\frac{z}{2}}{\sin^4\frac{z}{2}} = -\frac{1}{180}(\gamma^2 - 1)(7\gamma^2 + 17) \quad , \tag{B.13}$$

where $\gamma = \frac{2\pi}{\beta}$ (or, in general case, $\gamma = \frac{\beta_H}{\beta}$). By making use of these relations and Eq. (B.4) we find

$$\mathrm{Tr}_{\tilde{\Sigma}} K_\beta(s) - \mathrm{Tr}_{\tilde{\Sigma}} K(s) \simeq \frac{1}{(4\pi s)^{D/2-1}} \left(A_{\beta,1} + s A_{\beta,2} + O(s^2)\right) \quad , \tag{B.14}$$

where $A_{\beta,k}$ are given by Eqs. (3.16),(3.17) and Table 1 for spinors. Formula (B.14) determines the difference between asymptotic expansions on regular and singular spaces. It is easy to see that $A_{\beta,k}$ are the corrections to the spinor heat coefficients from conical singularities.

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